Cauchy Product

The Cauchy product of two absolutely convergent series is an infinite series that converges to the product of the sum of each series.

E.g.; $\sum_{k}^{\infty} u_k = U$, $\sum_{k}^{\infty} v_k = V$

Then the Cauchy product of u_k + v_k = UV

Cauchy Product of u_k + v_k =

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} u_k v_{n-k}$$

Like polynomial product

$$(a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n)$$
$$\times (b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n)$$
$$a_0 b_0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + \cdots$$

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	a_0	a_1	a_2	$\cdots$	a_n
b_0	1	x	x^2		x^n
b_1	x	x^2		$\cdot$	
b_2	x^2		$\cdot^{\cdot^{\cdot}}$		
$\vdots$		$\cdot^{\cdot}$			
b_n	x^n				

Proof that Cauchy product converges to product.

The proof proceeds in 3 parts:

1) We show that for a convergent series, the average of its partial sums converges to the sum of the series.
2) We show that the Cauchy product is convergent.
3) We show that the average of the partial sums of the Cauchy product is equal to the product UV, & because of step 1, so does the Cauchy product series itself.

1) Average of partial sums converges to series sum.

$$S = \sum_{k=0}^{\infty} x_k \quad , \quad S_n = \sum_{k=0}^{n} x_k \quad , \quad \sigma_n = \frac{1}{n+1}\sum_{k=0}^{n} S_k \Rightarrow \sigma_n \to S$$

series x_n converges to S — partial sums — average of partial sums — average of partial sums converges to the sum of the series.

Note that σ_n is an infinite sequence. We want to show that this sequence converges to the sum S.

Start by subtracting S from σ_n:

$$\sigma_n - S = \left(\frac{1}{n+1}\sum_{k=0}^{n} S_k\right) - S\left(\frac{n+1}{n+1}\right) = \frac{1}{n+1}\sum_{k=0}^{n}(S_k - S)$$

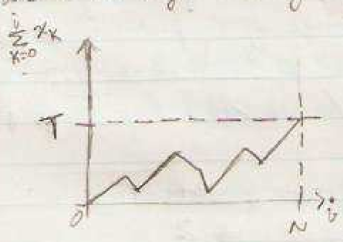
$\longrightarrow$

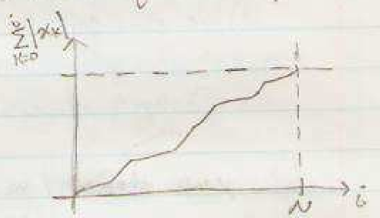
$$\sigma_n - S = \frac{1}{n+1}\sum_{k=0}^{n}(S_k - S)$$

Now lets look at what happens when we take the absolute values of the terms in the sum.

Imagine a sum in general: $T = \sum_{k=0}^{N} t_k$

If any of the values t_k are negative then the sum will move up & down as the terms are added up. If, instead we add up the absolute values of the terms: $\sum_{k=0}^{N}|t_k|$, the sum will increase monotonically as the terms are added up. That means the sum cannot be less than T, because we're no longer adding any negatives along the way.





Therefore

$$\sum_{k=0}^{N} x_k \le \sum_{k=0}^{N}|x_k|$$

Furthermore, the magnitude of T cannot exceed $\sum_{k=0}^{N}|x_k|$. To see this, consider how we can maximize the magnitude of the sum. We can do it one of two ways: by adding up all negative values, or by adding up all positive values. Either way, the magnitude of the sum is the same. But the latter case is the one we're already considering: $\sum_{k=0}^{N}|x_k|$, so in both cases, the magnitude of the sum is equal to $\sum_{k=0}^{N}|x_k|$, & therefore

$$|T| \le \sum_{k=0}^{N}|x_k|$$

Now that we know this:

$$\sigma_n - S = \frac{1}{n+1}\sum_{k=0}^{n}(S_k - S) \implies |\sigma_n - S| \leq \frac{1}{n+1}\sum_{k=0}^{n}|S_k - S|$$

Now, because S_k is the k-th partial sum of a series that sums to S, then the infinite sequence S_k converges to S. More precisely, for any value $\varepsilon > 0$, we can find some index M such that for all $k > M$, the difference between S_k & S is less than ε:

$$|S_k - S| < \varepsilon \text{ , for } k > M.$$

So we can now split our sum at M:

$$|\sigma_n - S| \leq \frac{1}{n+1}\left(\sum_{k=0}^{M}|S_k - S| + \sum_{k=M+1}^{n}|S_k - S|\right)$$

Looking at the second sum, since all of the terms are at indices greater than M, so the terms are all less than ε:

$$|\sigma_n - S| \leq \frac{1}{n+1}\left(\sum_{k=0}^{M}|S_k - S| + \sum_{k=M+1}^{n}\varepsilon\right) = \frac{1}{n+1}\sum_{k=0}^{M}|S_k - S| + \frac{n-M}{n+1}\varepsilon$$

Now lets look at the first sum, from $k=0$ to M. This sum is a fixed constant: no matter how large we let n get (how far we go in the sequence $|\sigma_n - S|$), M is fixed (by our choice of ε) so this finite sum is constant, which we call L:

$$L = \sum_{k=0}^{M}|S_k - S|$$

$$\therefore \quad |\sigma_n - S| \leq \frac{1}{n+1}L + \frac{n-M}{n+1}\varepsilon < \frac{L}{n+1} + \varepsilon$$

Where the second part follows because $\frac{n-M}{n+1}$ is always less than 1.

$\longrightarrow$

Repeating:

$$|\sigma_n - S| < \frac{L}{n+1} + \varepsilon$$

Now, we're trying to show that the infinite sequence σ_n converges to S. We're basically there w/ the inequality above.

We picked a value $\varepsilon > 0$, & found an appropriate index M, & that produced the constant L. So for any value $\varepsilon_2 > 0$, we can pick a value $\varepsilon < \varepsilon_2$ and then all we need to do is go far enough out on the sequence, say to some index N, such that $\frac{L}{N+1}$ is less than $\varepsilon_2 - \varepsilon$:

$$\frac{L}{N+1} + \varepsilon < \varepsilon_2$$

And therefore:

$$|\sigma_N - S| < \varepsilon_2$$

And so, we have shown that for any chosen limit, $\varepsilon_2 > 0$, we can choose an N such that the difference between σ_N & S is less than ε_2, & therefore σ_n converges to S. QED.

This completes part 1 of the Cauchy product proof.

Part 2 - Proof that Cauchy product converges.

Recall our definition of the Cauchy product:

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} u_k v_{n-k}$$

So if we define a sequence $c_k = \sum_{j=0}^{k} u_k v_{j-k}$, then the Cauchy product is $\sum_{n=0}^{\infty} c_n$.

So we want to show that $\sum_{n=0}^{\infty} c_n$ converges which means the Cauchy product converges.

We'll look at absolute convergence: $\sum_{n=0}^{\infty} |c_n|$. Recall that (as we showed a few pages ago, pg 79):

$$\left|\sum_{n=0}^{N} x_n\right| \leq \sum_{n=0}^{N} |x_n|$$

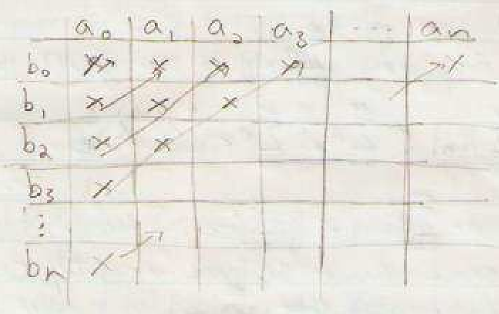
So since $c_n = \sum_{k=0}^{n} u_k v_{n-k}$, $|c_n| \leq \sum_{k=0}^{n} |u_k v_{n-k}|$, and so:

$$\sum_{n=0}^{\infty} |c_n| \leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} |u_k v_{n-k}|$$

And likewise $\sum_{n=0}^{\infty} c_n \leq \sum_{n=0}^{\infty} |c_n|$ so $\sum_{n=0}^{\infty} c_n \leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} |u_k v_{n-k}|$

For convenience, we will define $a_k = |u_k|$, + $b_k = |v_k|$.
Therefore $\sum_{n=0}^{\infty} c_n \leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} |u_k v_{n-k}| = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k}$.

The sum of these products traces a triangle in the a × b grid:



All the marked products are summed together, and the resulting sum is no less than the Cauchy product.

Clearly, if we sum all of the products in this square, it will not be less than the sum we just saw, which in turn is not less than the Cauchy Product:

$$\sum_{n=0}^{\infty}|c_n| \leq \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k b_{n-k} \leq \underbrace{\sum_{n=0}^{\infty}\left(a_n \sum_{k=0}^{\infty} b_k\right)}_{\text{sum of all the products}}$$

For instance: $(a_0b_0 + a_0b_1 + a_0b_2 + \cdots)$
$+ (a_1b_0 + a_1b_1 + a_1b_2 + \cdots)$
$+ \cdots$

and then you can pull the common a_k term out of each row:

$$a_0(b_0 + b_1 + b_2 + \cdots)$$
$$+ a_1(b_0 + b_1 + b_2 + \cdots)$$
$$+ \cdots$$

And now, we can pull the $(b_0 + b_1 + b_2 + \cdots)$ term out of it.

$$= (b_0 + b_1 + b_2 + \cdots)(a_0 + a_1 + a_2 + \cdots)$$

$$= \left(\sum_{k=0}^{\infty} b_k\right)\left(\sum_{k=0}^{\infty} a_k\right) = \text{sum of all the products in the square.}$$

$$\therefore \sum_{n=0}^{\infty}|c_n| \leq \sum_{k=0}^{\infty} a_k \sum_{k=0}^{\infty} b_k \leq A\,B$$

Where $A = \sum_{k=0}^{\infty} a_k$ & $B = \sum_{k=0}^{\infty} b_k$. Since $a_k = |u_k|$ and $b_k = |v_k|$, and since u_k & v_k are absolutely convergent, A + B are finite, and so $\sum_{n=0}^{\infty}|c_n|$ is bounded above, and therefore $\sum_{n=0}^{\infty} c_k$ (the Cauchy Product) is <u>absolutely convergent</u>. QED.

This concludes the second part of the proof.

Part 3 — Proof that the average of the partial sums of the Cauchy product converges to the product.

Recall that:

$$U = \sum_{k=0}^{\infty} u_k \ , \ V = \sum_{k=0}^{\infty} v_k$$

And we'll call the Cauchy product W:

$$W = \sum_{k=0}^{\infty} w_k \ , \quad w_k = \sum_{j=0}^{k} u_j v_{k-j}$$

And we have the partial sums:

$$U_n = \sum_{k=0}^{n} u_k \ ; \ V_n = \sum_{k=0}^{n} v_k \ ; \ W_n = \sum_{k=0}^{n} w_k = \sum_{k=0}^{n} \sum_{j=0}^{k} u_j v_{k-j}$$

We know, as a property of infinite series, that the sequence of partial sums converges to the sum of the series:

$$U_n \to U \ ; \ V_n \to V \ ; \ W_n \to W$$

We now wish to show that the Cauchy product, W, is equal to the product of the two series' sums:

$$W = UV$$

This is, after all, what we're trying to proove overall.

We start w/ the average the first $m+1$ partial sums of the Cauchy product:

$$\sigma_m = \frac{1}{m+1} \sum_{n=0}^{m} W_n$$

→

Now, as an intermediate, we're going to show that

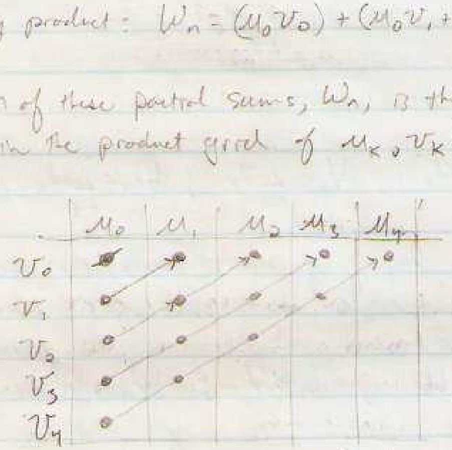
$$S_m := \sum_{n=0}^{m} W_n = \sum_{n=0}^{m} u_n v_{m-n}$$

So that we can rewrite our average σ_m in terms of u_k + v_k.

First let's look at W_n. This is a partial sum of the Cauchy product: $W_n = (u_0 v_0) + (u_0 v_1 + u_1 v_0) + \dots + (u_0 v_n + \dots + u_n v_0)$

So each of these partial sums, W_n, is the sum of $n+1$ diagonal rows in the product grid of u_k, v_k:

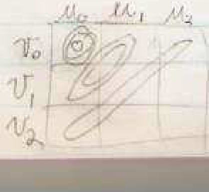
$W_4 =$



Marked products are summed to W_4. Each diagonal row is a full progression through the inner summation of the double summation of W_n.

So the first few partial sums look like:

Which means when we're adding up a bunch of these partial products together in S_m, we're actually adding the first diagonal $(u_0 v_0)$ $m+1$ times, + the second row $(u_0 v_1 + u_1 v_0)$ m times, etc:



Now we'll look at the right side: $\sum_{n=0}^{m} U_n V_{m-n}$

U_n is a partial sum of u_k: $U_0 = u_0$, $U_1 = u_0 + u_1$, $U_2 = u_0 + u_1 + u_2$, etc., & likewise with V_n, except that the partial sums of v_k are going in the other direction.

So this sum of products of partial sums looks like this:

$$\sum_{n=0}^{m} U_n V_{m-n} =$$

$(n=0,\ U_0 V_m)$ $\quad u_0 (v_0 + v_1 + \cdots + v_m)$

$(n=1,\ U_1 V_{m-1})$ $\quad + (u_0 + u_1)(v_0 + \cdots + v_{m-1})$

$(n=2)$ $\quad + (u_0 + u_1 + u_2)(v_0 + \cdots + v_{m-2})$

$\vdots$

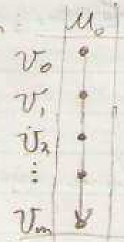
$(n=m-1)$ $\quad + (u_0 + \cdots + u_{m-1})(v_0 + v_1)$

$(n=m)$ $\quad + (u_0 + \cdots + u_m)(v_0)$

Each term, $U_n V_{m-n}$, in this sum also adds up some products in the u_k, v_k product grid. In this case, each term doesn't cover a triangle with sides of length m, but instead a rectangle of dimensions $(n+1) \times (m+1-n)$:

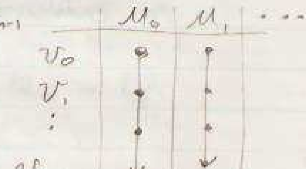
$n=0$: $U_0 V_m$

$1 \times (m+1)$



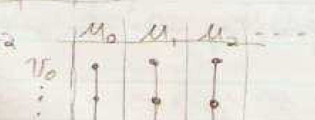
$n=1$: $U_1 V_{m-1}$

$2 \times m$



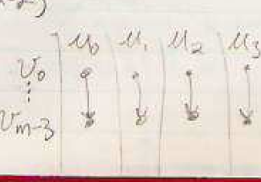
$n=2$: $U_2 V_{m-2}$

$3 \times (m-1)$



$n=3$: $U_3 V_{m-3}$

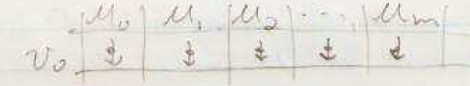
$4 \times (m-2)$



And finally:

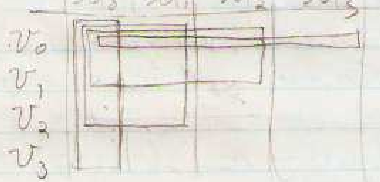
n=m: $U_m V_0$

$(m+1) \times 1$



So now we add all the rectangles together:

$\sum_{n=0}^{3} U_n V_{3-n}$:



And so we've got the same thing: the top left corner, $u_0 v_0$, is included in all $m+1$ rectangles. Each product in the second diagonal, $(u_0 v_1 + u_1 v_0)$ are each included in all but 1 rectangle ($u_0 v_1$ is in all but the last; $u_1 v_0$ is in all but the first), & so they're each summed m times. Each product in the third diagonal is included in all but 2 rectangles, & so are each included in the sum $m-1$ times, etc.

And so we see that

$$\sum_{n=0}^{m} W_n = \sum_{n=0}^{m} U_n V_{m-n}.$$

And we can therefore say that the average of our partial sums is

$$\sigma_m = \frac{1}{m+1} \sum_{n=0}^{m} U_n V_{m-n}$$

Now we define α_k + β_k to be the error terms between the kth partial sum + the series sum for u_k + v_k, respectively:

$$U_n = U + \alpha_n \quad , \quad V_n = V + \beta_n$$

And so:

$$\sigma_m = \frac{1}{m+1} \sum_{n=0}^{m} (U+\alpha_n)(V+\beta_{m-n})$$

$$= \frac{1}{m+1} \sum_{n=0}^{m} (UV + U\beta_{m-n} + V\alpha_n + \alpha_n\beta_{m-n})$$

$$= \frac{1}{m+1} UV \sum_{n=0}^{m} 1 + \frac{1}{m+1} U \sum_{n=0}^{m} \beta_{m-n} + \frac{1}{m+1} V \sum_{n=0}^{m} \alpha_n + \frac{1}{m+1} \sum_{n=0}^{m} \alpha_n \beta_{m-n}$$

$$= UV + \frac{U}{m+1} \sum_{n=0}^{m} \beta_n + \frac{V}{m+1} \sum_{n=0}^{m} \alpha_n + \frac{1}{m+1} \sum_{n=0}^{m} \alpha_n \beta_{m-n}$$

The first term comes because we're adding up $(m+1)$ 1's, + multiplying by $\frac{1}{m+1}$. Notice we changed β_{m-n} to β_n in the second term, That's ok, all we did was add up the series backwards.

So we're looking at an infinite sequence in σ_m. We know that the error terms, α_n + β_n, both go to zero as n goes to infinity, + so these two series: $\Sigma\alpha_n$ + $\Sigma\beta_n$, both converge, i.e., the sum is finite, Which means that the entire second + third terms can be driven arbitrarily close to zero by choosing a high enough m (going far enough out into the σ_m sequence).

Now we look at the final term $\frac{1}{m+1} \sum_{n=0}^{m} \alpha_n \beta_{m-n}$.

We know the the error terms are each bounded above, because the series of which they are the error terms converge. We'll call M the maximum of α_k + β_k for all k.

→

In otherwords:

$$|\alpha_k| < M, \ |\beta_k| < M \quad \text{for all } k.$$

Now, since α_k & β_k both converge to zero, we can also say that for any value $\varepsilon > 0$, there is some N such that:

$$|\alpha_k| < \varepsilon, \ |\beta_k| < \varepsilon, \quad \text{for } k > N.$$

Now we're going to show that this final term converges absolutely to zero. Note that this term is a sequence indexed by m:

$$\left| \frac{1}{m+1} \sum_{n=0}^{m} \alpha_n \beta_{m-n} \right| \leq \frac{1}{m+1} \sum_{n=0}^{m} |\alpha_n \beta_{m-n}|$$ (we showed this back in part 1).

Now we can split up our sum:

$$= \frac{1}{m+1} \left(\sum_{n=0}^{N} \alpha_n \beta_{m-n} + \sum_{n=N+1}^{m} \alpha_n \beta_{m-n} \right)$$

$$\leq \frac{1}{m+1} \left((N+1)(M^2) + (m-N)\,\varepsilon^2 \right)$$

$$= \frac{N+1}{m+1} M^2 + \frac{m-N}{m+1} \varepsilon^2 = \frac{N+1}{m+1} M^2 + \frac{m}{m+1} \varepsilon^2 - \frac{N}{m+1} \varepsilon^2$$

$$? \quad < \frac{N+1}{m+1} M^2 - \frac{N}{m+1} \varepsilon^2$$

Now N, M, & ε are all constants so this goes to zero as m goes to infinity, therefore that final term, $\frac{1}{m+1} \sum_{n=0}^{m} \alpha_n \beta_{m-n}$, converges absolutely to zero.

And so going back to the last page, we have

$$\sigma_m \to UV \quad \text{as } m \text{ goes to infinity. QED.}$$

This concludes the third & final part of the proof.

So in summary: The Cauchy product is absolutely convergent, & the average of its partial sums converges to the product UV. We showed in part 1 that the average of partial sums of an absolutely convergent series is equal to the sum of the series.

Therefore, the Cauchy Product is equal to UV. QED

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